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Small Data Existence for the Enskog Equation in L^1

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An existence theorem for the Enskog equation with small initial data is proved in an L^1 setting. This type of result is not available for the Boltzmann equation.

KEY WORDS: Boltzmann equation; kinetic theory; Enskog equation; existence.

1. INTRODUCTION

As is well known, the Boltzmann equation has so far resisted all attacks tending to prove a global existence theorem for sufficiently general data.(1) Several particular cases have been successfully dealt with, e.g., homogeneous problems,(2,3) near equilibrium solutions,(4–6) perturbation of a vacuum,(7–9) and, more recently, nearly homogeneous solutions(10) and homoenergetic affine flows.(11)

The difficulties appear to be related to the circumstance that the nonlinear collision term contains the product of the distribution function at one space point by the same function at the same space point (but for a different velocity argument). This fact was noticed long ago by Morgenstern,(12) who introduced a modification of the collision term by assuming two different arguments $\mathbf{x}$ and $\mathbf{x}_*$ for the two factors and an additional integration, thus producing an eightfold integration in the right-hand side of the Boltzmann equation. Later Povzner(13) indicated that a sixfold integration was sufficient, by taking the unit vector $\mathbf{n}$ appearing in the collision term to be directed along $\mathbf{x}-\mathbf{x}_*$ and adding an integration with respect to $|\mathbf{x}-\mathbf{x}_*|$. He even argued that this description is closer to

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physical reality than the one embodied in the Boltzmann equation. What is true, of course, is that two interacting mass points are separated at the moment of collision and their distance varies during the interaction; but the *rationale* of the Boltzmann equation is exactly that these and other effects are negligible for a dilute gas. Further, there is no indication that Povzner's equation, embodying these details in a purely formal way, is any closer to reality. In the particular case of rigid spheres, Povzner's argument is certainly wrong, because the distance between the centers is fixed in a collision and equal to the sphere diameter. Further, in this case a more accurate equation embodying the main effects differentiating a dense gas from a dilute one exists and was derived by Enskog as early as 1922.[14]

The Enskog equation retains the fivefold integration typical of the Boltzmann equation and differs from the latter because it takes into account (1) many-body effects, modifying the collision frequency, and (2) the different location of the centers of the spheres during a collision.

While the first of these effects should really become important for dense gases only, the second effect, in order to be negligible even for a dilute gas, requires the distribution function to be continuous (in some sense) on the scale of the sphere diameter σ. It is thus of some importance to investigate the Enskog equation embodying factor 2 and later try, if possible, to take the limit when σ goes to zero. This investigation should also shed light on the problem of the derivation of the Boltzmann equation from molecular dynamics.[15–19]

The fact that the Enskog equation has only a fivefold integral implies that Povzner's argument[13] does not work. It was recently shown,[20] however, that an existence proof can be constructed if the data depend on just one space variable. In a subsequent paper[21] (which, however, appeared earlier) the case of data depending on two space variables was attacked; the Enskog equation, however, was modified by including non-physical collisions.

In this paper the problem of existence of solutions of the Enskog equation with initial data depending on all the three space variables is attacked. The effects related to the modification of the collision frequency at high densities (denoted by factor 1 above) are omitted, as was done in Refs. 20 and 21.

Global existence and continuous dependence on initial data are proved in L^1 for sufficiently small data. We remark that no such result is available for the Boltzmann equation. In fact, all the small-data results[7–9] refer to function spaces of the L^∞ type. Indeed, if an L^1 result were available for the Boltzmann equation, one could presumably obtain a result for arbitrarily large data, by first cutting the high speeds, exploiting hyperbolicity and the H-theorem, and then passing to the limit of no cutoff.

This program is not so easy to carry out for the Enskog equation, because strict hyperbolicity is lost even with the velocity cutoff, because of the displaced arguments in the collision term.

2. BASIC EQUATIONS AND PRELIMINARY RESULTS

The Enskog equation to be considered in this paper reads as follows:

$$\frac{\partial f}{\partial t}+\xi\cdot\frac{\partial f}{\partial x}=Q(f,f) \tag{2.1}$$

where

$$Q(f,f)=\sigma^2\iiint(f'f'_*-ff_*)\,|\mathbf{V}\cdot\mathbf{n}|\,H(\mathbf{V}\cdot\mathbf{n})\,d\xi_*\,d\mathbf{n} \tag{2.2}$$

Here the arguments of f_*, f', and f'_* are $(\mathbf{x}_*,\xi_*,t)$, $(\mathbf{x},\xi',t)$, and $(\mathbf{x}_*,\xi'_*,t)$ rather than $(\mathbf{x},\xi,t)$, and H denotes, as usual, Heaviside's step function. Further,

$$\begin{aligned}\mathbf{V}&=\xi-\xi_*\\ \xi'&=\xi-\mathbf{n}(\mathbf{n}\cdot\mathbf{V})\\ \xi'_*&=\xi_*+\mathbf{n}(\mathbf{n}\cdot\mathbf{V})\\ \mathbf{x}_*&=\mathbf{x}+\mathbf{n}\sigma\end{aligned} \tag{2.3}$$

$\mathbf{n}$ ranges over a unit sphere, or rather, because of the Heaviside step function, over a half of such a sphere.

Equation (2.1) is to be solved with the initial data

$$f(\mathbf{x},\boldsymbol{\sigma},0)=\phi(\mathbf{x},\xi) \tag{2.4}$$

and the condition that f is in $L^1(R)$ for any ξ.

We shall now prove some preliminary results. To this end, we fix an arbitrarily large time interval $I_T=[0,T]$. Then we introduce the following function space:

$$F=\left\{f(\mathbf{x},\xi,t)\text{ defined in }D\text{ with }\frac{\partial f}{\partial t}+\xi\cdot\frac{\partial f}{\partial \mathbf{x}}\in L^1(D\times R^3);\right.$$
$$\left.\phi(\mathbf{x},\xi)=f(\mathbf{x},\xi,0)\in L^1(R^3\times R^3)\right\} \tag{2.5}$$

where

$$D=R^3\times I_T \tag{2.6}$$

with norm

$$\|f\|_F = \|\phi\|_{L^1(R^3\times R^3)} + \left\| \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial \mathbf{x}} \right\|_{L^1(D\times R^3)} \tag{2.7}$$

F is a Banach space isometric to $L^1(D\times R^3)\times L^1(R^3\times R^3)$.

The first result is contained in the following result, analogous to one proved by Tartar[22] for solutions of discrete velocity models depending on just one space variable:

Lemma 2.1. If $f\in F$, then there is a $g\in L^1(R^3\times R^3)$ such that

$$|f(\mathbf{x}, \xi, t)| \leqslant g(\mathbf{x}-\xi t, \xi) \qquad \text{a.e.} \quad \text{in } D \tag{2.8}$$

$$\|g\|_{L^1(R^3\times R^3)} = \|f\|_F \tag{2.9}$$

Proof. Let

$$Q(\mathbf{x}, \xi, t) = \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial \mathbf{x}} \tag{2.10}$$

Then

$$f(\mathbf{x}, \xi, t) = \phi(\mathbf{x}-\xi t, \xi) + \int_0^t Q(\mathbf{x}-\xi t+\xi s, \xi, s)\, ds \tag{2.11}$$

Define

$$g(\mathbf{x}, \xi) = |\phi(\mathbf{x}, \xi)| + \int_0^T |Q(\mathbf{x}+\xi s, \xi, s)|\, ds \tag{2.12}$$

Then Eq. (2.11) gives Eq. (2.8). Equation (2.9) follows trivially from the definition of the norm in F, Eq. (2.7).

The second result is given by the following:

Lemma 2.2. If $f\in F$, then $Q(f, f)\in L^1(D\times R^3)$ and

$$\|Q(f, f)\|_{L^1(D\times R^3)} \leqslant \|f\|_F^2 \tag{2.13}$$

Proof. By Lemma 2.1 it is enough to bound

$$\begin{aligned}\sigma^2 \iiiint\!\!\int & [g(\mathbf{x}-\xi' t, \xi')\, g(\mathbf{x}+\sigma\mathbf{n}-\xi'_* t, \xi'_*) \\ & + g(\mathbf{x}-\xi t, \xi)\, g(\mathbf{x}+\sigma\mathbf{n}-\xi_* t, \xi_*)] \\ & \times |\mathbf{V}\cdot\mathbf{n}|\, H((\xi-\xi_*)\cdot\mathbf{n})\, d\xi_*\, d\mathbf{n}\, d\xi\, dt\, d\mathbf{x}\end{aligned}$$

$$= \sigma^2 \iiiint\!\!\int g(\mathbf{y}, \xi)\, g(\mathbf{y} + (\xi - \xi_*)t + \sigma \mathbf{n}, \xi_*)$$

$$\times |\mathbf{V} \cdot \mathbf{n}|\, (H((\xi - \xi_*) \cdot \mathbf{n}) + H(-(\xi - \xi_*) \cdot \mathbf{n}))\, d\xi_*\, d\xi\, d\mathbf{y}\, d\mathbf{n}\, dt$$

$$= \iiiint g(\mathbf{y}, \xi)\, g(\mathbf{z}, \xi_*)\, d\mathbf{y}\, d\mathbf{z}\, d\xi\, d\xi_*$$

$$\leqslant \|g\|^2_{L^1(R^3 \times R^3)} \tag{2.14}$$

where the usual change of variables $(\xi', \xi'_*) \to (\xi, \xi_*)$ has been performed and

$$\mathbf{y} = \mathbf{x} - \xi t, \qquad \mathbf{z} = \mathbf{y} + (\xi - \xi_*)t + \mathbf{n}\sigma \tag{2.15}$$

The integral with respect to $\mathbf{z}$ is extended to a subset of R^3. Use has been made of the fact that the Jacobian of $\mathbf{z}$ with respect to $(t, \mathbf{n})$ is $|\mathbf{V} \cdot \mathbf{n}|$.

Equation (2.14) shows that the statement of the lemma holds, since Eq. (2.13) follows from Eq. (2.14) and Lemma 2.1.

Lemma 2.3. The following inequality holds:

$$\sup_{0 \leqslant t \leqslant T} \|f\|_{L^1(R^3 \times R^3)} \leqslant \|f\|_F \tag{2.16}$$

In fact, f is given by Eq. (2.11) [where Q is defined in Eq. (2.10)]. Integrating Eq. (2.11) over $R^3 \times R^3$ and using Eq. (2.7) gives Eq. (2.16).

3. GLOBAL EXISTENCE FOR SMALL L^1 DATA

We are now ready to prove the following:

Theorem 3.1. There is a constant C_0 such that if $\phi \in L^1(R^3 \times R^3)$ and satisfies

$$C \equiv \|\phi\|_{L^1(R^3 \times R^3)} \leqslant C_0 \tag{3.1}$$

then there exists a unique solution to Eq. (2.9) [with $Q_c(f, f)$ in place of $Q(f, f)$] for any time interval I_T. This solution satisfies

$$\left\| \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial \mathbf{x}} \right\|_{L^1(R^3 \times R^3 \times I_T)} \leqslant \frac{1}{3} \|\phi\|_{L^1} \tag{3.2}$$

In order to prove this result, we use the contraction mapping theorem. Accordingly, we construct a mapping $h \to f = N(h)$ from F into itself in the following way.

Given $\phi \in L^1$ and $h \in F$, f is the solution of

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial \mathbf{x}} = Q(h, h) \tag{3.3}$$

$$f(\mathbf{x}, \xi, 0) = \phi(\mathbf{x}, \xi) \tag{3.4}$$

By Lemma 2.2 we have

$$\|Q(h, h)\|_{L^1(D \times R^3)} \leqslant \|h\|_F^2 \tag{3.5}$$

A mapping $h \to f = N(h)$ is thus established. We want to see that by suitably choosing C_0 in Eq. (3.1), this mapping is a contraction on some closed set of F.

In fact, Eqs. (3.3)–(3.5) give

$$\left\| \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \xi \right\|_{L^1(D \times R^3)} \leqslant \|h\|_F^2 \tag{3.6}$$

and hence

$$\|f\|_F \leqslant \|\phi\|_{L^1(D)} + \|h\|_F^2 = C + \|h\|_F^2 \tag{3.7}$$

Then N maps the ball B_R defined by $\|h\|_F \leqslant R$ into another ball $B_{\hat{R}}$ if

$$\hat{R} \leqslant C + R^2 \tag{3.8}$$

We can now bound the Lipschitz constant of N on B_R. If $\bar{h} \in B_R$ and $\bar{f} = N(\bar{h})$, then

$$\|f - \bar{f}\|_F \leqslant \|h + \bar{h}\|_F \, \|h - \bar{h}\|_F \leqslant 2R \, \|h - \bar{h}\|_F \tag{3.9}$$

Now, if we define C_0 to be a constant less than 3/16, we have a strict contraction if $R = 1/4$ and the theorem is proved. In particular, Eq. (3.7) with $h = f$ gives

$$\|f\|_F \leqslant C + \tfrac{1}{4}\|f\|_F \tag{3.10}$$

and Eq. (3.2) follows.

We can now prove a simple result on the continuous dependence on the initial data.

Theorem 3.2. If two initial data $\phi \in L^1(R^3 \times R^3)$ and $\bar{\phi} \in L^1(R^3 \times R^3)$ satisfy Eq. (3.1), then the corresponding solutions satisfy

$$\sup_{t \in R} \|f - \bar{f}\|_{L^1(R^3 \times R^3)} \leqslant \tfrac{4}{3}\|\phi - \bar{\phi}\|_{L^1(R^3 \times R^3)} \tag{3.11}$$

It is sufficient to note that Eq. (3.9) with $h=f$ can be modified to cover the case of different initial data, with the following result:

$$\|f-\bar{f}\|_F \leqslant \|\phi-\bar{\phi}\|_{L^1(R^3\times R^3)} + \tfrac{1}{4}\|f-\bar{f}\|_F \tag{3.12}$$

Hence

$$\|f-\bar{f}\|_F \leqslant \tfrac{4}{3}\|\phi-\bar{\phi}\|_{L^1(R^3\times R^3)} \tag{3.13}$$

and Eq. (3.11) holds, thanks to Lemma (2.3).

Note Added. I thank the referee for calling my attention to Ref. 23 by Lachowicz, which proves local existence and uniqueness in L^1, and Ref. 24 by Bellomo and Lachowicz, which proves global existence for near vacuum data.

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